RECURTION OPERATOR AND RATIONAL LAX REPRESENTATION

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Abstract

We consider equations arising from rational Lax representations. A general method to construct recursion operators for such equations is given. Several examples are given, including a degenerate bi-Hamiltonian system with a recursion operator.

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I. Introduction

Recently a new method of constructing a recursion operator from Lax representation was introduced in [1]. This construction depends on Lax representation of a given system of PDEs. Let

$$L_t = [A, L] \tag{1}$$

be Lax representation of an integrable nonlinear system of PDEs. Then a hierarchy of symmetries can be given by

$$L_{t_n} = [A_n, L], \qquad n = 0, 1, 2 \ldots, \tag{2}$$

where $t_0 = t$, $A_0 = A$ and $A_n, \quad n = 0, 1, 2 \ldots,$ are Gel'fand-Dikkii operators given in terms of L. The recursion relation between symmetries can be written as

$$L_{t_{n+1}} = LL_{t_n} + [R_n, L], \qquad n = 0, 1, 2 \ldots, \tag{3}$$

where R_n is an operator such that $ordR_n = ordL$.
This symmetry relation allows to find R_n, hence $L_{t_{n+1}}$, in terms of L and L_{t_n}.

In [1], [2] this method was applied to construct recursion operators for Lax equations with different classes of scalar and shift operators, corresponding to field and lattice systems respectively. In [3] the method was applied to Lax equations on a Poissson algebra of Laurent series

$$\Lambda = \left\{ \sum_{-\infty}^{+\infty} u_i p^i \ : u_i - \text{smooth} \quad \text{functions} \right\} \tag{4}$$

with the polynomial Lax function. Such equations give systems of hydrodynamic type. They were also discussed in [4]– [7]. The Hamiltonian structure of the Lax equation on a Poisson algebra was studed in [8].

Here we consider the Lax equation on the Poisson algebra Λ with a rational Lax function

$$L = \frac{\Delta_1}{\Delta_2} \ , \tag{5}$$

where Δ_1, Δ_2 are polynomials of degree N and M, respectively, and $N > M$. The Lax equation is

$$\frac{\partial L}{\partial t_n} = \{(L)_{\geq 0}^{\frac{1}{N-M}+n}, L\} \ , \tag{6}$$

where the Poisson bracket is given by

$$\{f, g\} = p\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p}\right) .$$

First we study the symmetry relation (3) for the rational Lax function. Then we give some examples.

In particular, we find a recursion operator $\mathcal{R}$ for equation (6) with the Lax function

$$L = p + S + \frac{P}{p+Q} \ , \tag{7}$$

which leeds to the system [4]

$$\begin{aligned} S_t &= P_x, \\ P_t &= PS_x - QP_x - PQ_x, \\ Q_t &= QS_x - QQ_x. \end{aligned} \tag{8}$$

The recursion operator is given by

$$\mathcal{R} = \begin{pmatrix} S & 1 & PQ^{-1} + P_x D_x^{-1} \cdot Q \\ 2P & S - Q & -2P + (PS_x - (PQ)_x) D_x^{-1} \cdot Q \\ Q & 1 & PQ^{-1} + S - Q + (QS_x - QQ_x) D_x^{-1} \cdot Q \end{pmatrix} \tag{9}$$

In [4] bi-Hamiltonian representation of this equation was constructed with Hamiltonian operators

$$\mathcal{D}_1 = \begin{pmatrix} 0 & P & Q \\ P & -2PQ & -Q^2 \\ Q & -Q^2 & 0 \end{pmatrix} D_x + \begin{pmatrix} 0 & P_x & Q_x \\ 0 & -(PQ)_x & -QQ_x \\ 0 & -QQ_x & 0 \end{pmatrix} \tag{10}$$

and

$$\mathcal{D}_2 = \begin{pmatrix} 2P & P(S-3Q) & Q(S-Q) \\ P(S-3Q) & P(2P - 2SQ + 4Q^2) & Q(2P - SQ + Q^2) \\ Q(S-Q) & Q(2P - SQ + Q^2) & 2Q^2 \end{pmatrix} D_x + \tag{11}$$

$$\begin{pmatrix} P_x & SP_x - 2(PQ)_x & SQ_x - QQ_x \\ PS_x - (QP)_x & (-SPQ + P^2 + 2PQ^2)_x & Q_x(2P + Q^2 - SQ) \\ QS_x - QQ_x & Q(2P_x + 2QQ_x - S_x - SQQ_x) & 2QQ_x \end{pmatrix}$$

These Hamiltonian operators are degenerate, so, one can not use them to find a recursion operator. But it turns out that they are related to the recursion operator $\mathcal{R}$. One can easily check that the following equality holds

$$\mathcal{R}\mathcal{D}_1 = \mathcal{D}_2.$$

We observe that the degeneracy in the bi-Hamiltonian operators is due to the following fact. Let $p' = p + F$ then the Lax function becomes

$$L = p' + G + \frac{P}{p'} \ . \tag{12}$$

This means that we have two independent variabels P and G, where $G = S - F$. The equation corresponding to the Lax function (12) has been studied in [3].

To remove degeneracy one can take the Lax function as

$$L = p + S + \frac{P}{p} + \sum_{i=1}^{m} \frac{Q_i}{p + F_i} \ . \tag{13}$$

As an example we shall consider the equation (6) with the Lax function

$$L = p + S + \frac{P}{p} + \frac{Q}{p + F} \ . \tag{14}$$

II. Symmetry Relation for Rational Lax Representation.

Following [1] we consider the hierarchy of symmetries for the Lax equation (6) with the Lax function (5)

$$\frac{\partial L}{\partial t_n} = \{(L^{\frac{1}{N-M}+n})_{\geq 0}, L\}. \tag{15}$$

Lemma 1. *For any* $n = 0, 1, 2, \ldots,$

$$\frac{\partial L}{\partial t_n} = L \frac{\partial L}{\partial t_{n-1}} + \{R_n, L\}. \tag{16}$$

Function R_n has a form

$$R_n = A + \frac{B}{\Delta_2} \tag{17}$$

where A is a polynomial of degree $(N-M)$ and B is a polynomial of degree $(M-1)$.

Proof. We have

$$(L^{\frac{1}{N-M}+n})_{\geq 0} = [L(L^{\frac{1}{N-M}+(n-1)})_{\geq 0} + L(L^{\frac{1}{N-M}+(n-1)})_{<0}]_{\geq 0}$$

So,

$$(L^{\frac{1}{N-M}+n})_{\geq 0} = L(L^{\frac{1}{N-M}+(n-1)})_{\geq 0} + (L(L^{\frac{1}{N-M}+(n-1)})_{<0})_{\geq 0} - (L(L^{\frac{1}{N-M}+(n-1)})_{\geq 0})_{<0}.$$

If we take

$$R_n = (L(L^{\frac{1}{N-M}+(n-1)})_{<0})_{\geq 0} \ - (L(L^{\frac{1}{N-M}+(n-1)})_{\geq 0})_{<0} \quad , \tag{18}$$

then

$$(L^{\frac{1}{N-M}+n})_{\geq 0} = L(L^{\frac{1}{N-M}+(n-1)})_{\geq 0} \ + R_n.$$

Hence,

$$\frac{\partial L}{\partial t_n} = \left\{(L^{\frac{1}{N-M}+n})_{\geq 0}; L\right\} = \left\{L(L^{\frac{1}{N-M}+(n-1)})_{\geq 0} + R_n; L\right\} = L\frac{\partial L}{\partial t_n} + \{R_n; L\},$$

and (16) is satisfied. The remainder R_n has form (17). Indeed in (18) we set

$$A = (L(L^{\frac{1}{N-M}+(n-1)})_{<0})_{\geq 0}$$

and

$$B = \Delta_2 \cdot (L(L^{\frac{1}{N-M}+(n-1)})_{\geq 0})_{<0}$$

Then A is a polynomial of degree $(N - M - 1)$ and B is a polynomial of degree $(M - 1)$. $\square$

Now we can apply the Lemma 1 to find recursion operators.

III. Examples.

Example 2. Let us consider the equation (8) given in introduction.

Lemma 3. *A recursion operator for (8) is given by (9).*

Proof. Using (17) for R_n, we have $R_n = A + \dfrac{B}{p+Q}$. So, the symmetry relation (16) is

$$\frac{\partial S}{\partial t_n} + \frac{\partial P}{\partial t_n} \cdot \frac{1}{p+Q} + \frac{\partial Q}{\partial t_n} \cdot \frac{P}{(p+Q)^2} =$$

$$\left(p + S + \frac{P}{p+Q}\right)\left(\frac{\partial S}{\partial t_{n-1}} + \frac{\partial P}{\partial t_{n-1}} \cdot \frac{1}{p+Q} + \frac{\partial Q}{\partial t_{n-1}} \cdot \frac{P}{(p+Q)^2}\right) +$$

$$p\left(A_x + \frac{B_x}{p+Q} + \frac{-BQ_x}{(p+Q)^2}\right)\left(1 + \frac{-P}{(p+Q)^2}\right)$$

$$-\frac{pB}{(p+Q)^2}\left(S_x + \frac{P_x}{p+Q} + \frac{-PQ_x}{(p+Q)^2}\right)$$

To have the equality the coefficients of p and $(p+Q)^{-3}$ must be zero. It gives the recursion relations to find A and B. Then the coefficients of p^0, $(p+Q)^{-1}$, $(p+Q)^{-2}$ give expressions for $\dfrac{\partial S}{\partial t_n}$, $\dfrac{\partial P}{\partial t_n}$, $\dfrac{\partial Q}{\partial t_n}$. $\square$

Example 4. The Lax equation (6) with the Lax function (14), for $n = 1$, gives the following system

$$
\begin{aligned}
S_t &= P_x + Q_x, \\
P_t &= PS_x, \\
Q_t &= QS_x - FQ_x - QF_x, \\
F_t &= FS_x - FF_x.
\end{aligned} \tag{19}
$$

Lemma 5. *A recursion operator for (19) is given by*

$$
\begin{pmatrix}
S & 2 + P_x D_x^{-1} \cdot P^{-1} & 1 & QF^{-1} + Q_x D_x^{-1} \cdot F^{-1} \\
 & & & \\
2P & \begin{array}{l} S + QF^{-1} + PS_x D_x^{-1} \cdot P^{-1} \\ PF^{-1}(Q_x - QF^{-1}F_x) D_x^{-1} \cdot P^{-1} \end{array} & PF^{-1} & \begin{array}{l} -2PQF^{-2} \\ -PF^{-1}(Q_x - QF^{-1}F_x) D_x^{-1} \cdot F^{-1} \end{array} \\
 & & & \\
2Q & \begin{array}{l} -QF^{-1} \\ -PF^{-1}(Q_x - QF^{-1}F_x) D_x^{-1} \cdot P^{-1} \end{array} & \begin{array}{l} S - F \\ -PF^{-1} \end{array} & \begin{array}{l} -2PQF^{-2} - 2Q \\ +PF^{-1}(Q_x - QF^{-1}F_x) D_x^{-1} \cdot F^{-1} \\ +(QS_x - QF_x - FQ_x) D_x^{-1} \cdot F^{-1} \end{array} \\
 & & & \\
F & 1 + (P_x - PF^{-1}F_x) D_x^{-1} \cdot P^{-1} & -1 & \begin{array}{l} PF^{-1} - F + (FS_x - FF_x) D_x^{-1} \cdot F^{-1} \\ -(P_x - PF^{-1}F_x) D_x^{-1} \cdot F^{-1} \end{array}
\end{pmatrix}. \tag{20}
$$

Proof. Using (17) for R_n, we have $R_n = C + \dfrac{A}{p} + \dfrac{B}{p+F}$. So, the symmetry relation (16) is

$$
\frac{\partial S}{\partial t_n} + \frac{\partial P}{\partial t_n} \cdot \frac{1}{p} + \frac{\partial Q}{\partial t_n} \cdot \frac{1}{(p+F)} + \frac{\partial F}{\partial t_n} \cdot \frac{-Q}{(p+F)^2} =
$$

$$
\left(p + S + \frac{P}{p} + \frac{Q}{p+F}\right)\left(\frac{\partial S}{\partial t_{n-1}} + \frac{\partial P}{\partial t_{n-1}} \cdot \frac{1}{p} + \frac{\partial Q}{\partial t_{n-1}} \cdot \frac{1}{(p+F)} + \frac{\partial F}{\partial t_{n-1}} \cdot \frac{-Q}{(p+F)^2}\right) +
$$

$$p\left(\frac{-B}{p^2}+\frac{-C}{(p+F)^2}\right)\left(S_x+\frac{P_x}{p}+\frac{Q_x}{(p+F)}+\frac{-QF_x}{(p+F)^2}\right)-$$
$$p\left(A_x+\frac{B_x}{p}+\frac{C_x}{(p+F)+\frac{-CF_x}{(p+F)^2}}\right)\left(1+\frac{P}{p}+\frac{-Q}{(p+F)^2}\right)$$

Therefore, the coefficients of p, p^{-2}, and $(p+F)^{-3}$ must be zero, it gives recursion relations to find A, B and C. Then the coefficients of p^0, p^{-1}, $(p+F)^{-1}$ and $(p+F)^{-2}$, give expressions for $\frac{\partial S}{\partial t_n}$, $\frac{\partial P}{\partial t_n}$, $\frac{\partial Q}{\partial t_n}$ and $\frac{\partial F}{\partial t_n}$. $\square$

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